Two-Dimensional Radiating Gas Flow by a Moment Method

PING CHENG*
Stanford University, Stanford, Calif.

Introduction

THE governing equations for multidimensional radiating gas flow, in general, are of a complicated integrodifferential form. Thus, no reasonably simple solutions, valid for the whole range of optical thickness, exist for such problems at the present time. For the limiting cases of an optically thin or thick gas, the integrodifferential equations can be reduced to a system of differential equations, and earlier investigations[1-3] have been confined to these limits. For one-dimensional flow problems, Vincenti and Baldwin[4] suggested the use of an exponential function to approximate the integral term. By this approximation, they were able to obtain a system of purely differential equations that are valid for the whole range of optical thickness. Unfortunately, this technique is not directly applicable to multidimensional flow problems.

A different scheme of approximation known as the moment method was first suggested by astrophysicists[5] in problems of transfer of radiation in stars. It was subsequently highly developed by nuclear physicists[6] in dealing with neutron transport problems. Using this scheme of approximation, Traugott[7] and others[8] were able to obtain a system of purely differential equations for one-dimensional flow problems that are valid throughout the whole range of optical thickness. It can be shown that the first approximation in this approach when applied to one-dimensional transfer problems is equivalent to the exponential approximation suggested by Vincenti and Baldwin. In the following, we will apply this technique to multidimensional radiating-gas-flow problems.

Basic Equations

We begin by writing down the hydrodynamic equations for the three-dimensional time-dependent flow. The equations for conservation of mass, momentum, and energy in indicial notation are, respectively,

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_j}{\partial x_j} = 0 \tag{1}$$

$$\frac{Du_i}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_1} = 0 \qquad (i = 1,2,3) \tag{2}$$

$$\rho\frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{\partial q_j}{\partial x_j} = 0 \tag{3}$$

where p, h, ρ, u_i, q_i are, respectively, the pressure, enthalpy, density, velocity components, and radiation heat-flux components. In these equations, it is assumed that the effects of viscosity and conductivity are negligible and that the radiation-energy density and radiation pressure are small compared with the internal energy and pressure of the flow.

If all nonequilibrium effects other than radiation are neglected, the thermal and caloric equations of state for an assumed perfect gas are

$$p = \rho RT \tag{4}$$

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} \tag{5}$$

where T is the temperature and γ is the specific-heat ratio, which is assumed to be constant.

If we further assume that the gas is in local thermodynamic equilibrium, the radiation-transport equation for a gray gas as given by Chandrasekhar[5] is

$$\frac{1}{c}\frac{\partial I}{\partial t} + l_j\frac{\partial I}{\partial x_j} = -\alpha I + \frac{\alpha\sigma T^4}{\pi} \tag{6}$$

where α is the absorption coefficient, c the speed of light, σ the Stefan-Bolzmann constant, T the temperature of the gas, $I(\mathbf{r},\mathbf{\Omega})$ the specific radiation intensity, which is a function of the position vector $\mathbf{r}$ and the directional vector $\mathbf{\Omega}$, and the l_i's are the direction cosines of vector $\mathbf{\Omega}$ with respect to coordinates x_i. Owing to the magnitude of c, the first term in Eq. (6) is always much smaller than other terms and therefore can be neglected even for time-dependent situations.

The radiation heat flux and the radiation intensity are related by the integral

$$q_i(\mathbf{r}) = \int_\Omega I(\mathbf{r},\mathbf{\Omega})l_i d\Omega \tag{7}$$

where $d\Omega$ is an element of solid angle.

Equations (1–7) are the governing equations for radiating gas flow. If we substitute Eq. (7) into Eq. (3), it is readily seen that the governing equations are of integrodifferential form. Exact solutions for multidimensional radiating-gas-flow problems are therefore difficult, if not impossible, to obtain.

Approximate Radiation-Transport Equations

The moment method has been widely used in neutron-transport theory to approximate the exact transport equation by a finite number of moment equations. In the following, we shall use this technique as applied to multidimensional radiating-gas-flow problems.† To this end, we first introduce the moments of intensity as follows:

$$\left.\begin{aligned} I_0(\mathbf{r}) &\equiv \int I(\mathbf{r},\mathbf{\Omega})d\Omega \\ I_{1x_i}(\mathbf{r}) &\equiv \int I(\mathbf{r},\mathbf{\Omega})l_i d\Omega \\ I_{2x_ix_j}(\mathbf{r}) &\equiv \int I(\mathbf{r},\mathbf{\Omega})l_il_j d\Omega \\ &\cdot \\ &\cdot \\ &\cdot \\ I_{nx_ix_j}(\mathbf{r}) &\equiv \int I(\mathbf{r},\mathbf{\Omega})l_i(l_j)^{n-1}d\Omega \text{ for } n \geq 2 \end{aligned}\right\} \tag{8}$$

The first three moments have physical significance. This can be seen by noting that the radiation energy density u_r, heat flux vector q_i, and radiation pressure tensor p_{ij}, are given by

$$u_r(\mathbf{r}) = (1/c)\int_\Omega I(\mathbf{r},\mathbf{\Omega})d\Omega \tag{9a}$$

$$q_i(\mathbf{r}) = \int_\Omega I(\mathbf{r},\mathbf{\Omega})l_i d\Omega \tag{9b}$$

$$p_{ij}(\mathbf{r}) = (1/c)\int_\Omega I(\mathbf{r},\mathbf{\Omega})l_il_j d\Omega \tag{9c}$$

For the special case when the gas is in radiation equilibrium, Eqs. (9a) and (9c) immediately yield the relation

$$p_{ij} = (u_r/3)\,\delta_{ij} \tag{10}$$

where δ_{ij} is the Kronecker delta. It follows from Eqs. (8) and (9) that

$$u_r(\mathbf{r}) = I_0(\mathbf{r})/c \tag{11a}$$

$$q_i(\mathbf{r}) = I_{1x_i}(\mathbf{r}) \tag{11b}$$

$$p_{ij}(\mathbf{r}) = I_{2x_ix_j}(r)/c \tag{11c}$$

Received May 13, 1964. The author wishes to express his gratitude to W. G. Vincenti for his enlightening discussions, encouragement, and critical reading of the manuscript. The work here presented was supported by the U. S. Air Force Office of Scientific Research under Contract AF49(638)1280.

* Research Assistant, Department of Aeronautics and Astronautics.

† The procedure we follow here is essentially a generalization of the one-dimensional moment method presented by Krook,[9] Eddington,[5] and Traugott.[7] An alternative treatment of the moment method can be found in Ref. 6.

Thus, the first three moments are proportional to the radiation energy density, radiation heat flux, and radiation pressure tensor, respectively. Integrating Eq. (6) (with $\partial I/c\partial t$ neglected) over the complete range of solid angle and noting that T is independent of $\mathbf{\Omega}$, we have

$$\partial I_{1x_j}/\partial x_j = -\alpha I_0 + 4\alpha\sigma T^4 \qquad (12a)$$

Similarly, multiplying Eq. (6) by various powers of the direction cosines l_i ($i = 1,2,3$) and integrating, we also have

$$\partial I_{2x_ix_j}/\partial x_j = -\alpha I_{1x_i} \qquad (i = 1,2,3) \qquad (12b)$$

$$\vdots$$

The infinite set of moment equations (12) and the exact transport equation (6) are completely equivalent. We can, however, use a finite number of these equations to approximate the exact equation. In this case, the number of unknowns are usually more than the number of equations. In order to obtain a determinate set of equations for the nth approximation, we have to express the nth moment in terms of lower moments. This can be done as follows. Let the radiation intensity be expanded in a series in terms of normalized spherical harmonics:

$$I(\mathbf{r},\mathbf{\Omega}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l^m(\mathbf{r})Y_l^m(\mathbf{\Omega}) \qquad (13a)$$

where A_l^m's are functions to be determined and $Y_l^m(\Omega)$'s are the normalized spherical harmonics, each of which being related to the associated Legendre function $P_l^m(\cos\theta)$ by

$$Y_l^m(\mathbf{\Omega}) = \left[\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} e^{im\phi}\,P_l^m\,(\cos\theta) \qquad (13b)$$

The nth approximation follows when we terminate this series after the term with $l = 2n - 1$. It is shown in neutron-transport theory[6] that the first approximation is sufficiently accurate for a number of problems. For our purpose, therefore, we shall limit ourselves for the present to the first approximation. To this end, we let $A_l^m = 0$, for $l \geq 2$. If we substitute the truncated series into the first three moments of the intensity and use the orthogonal properties of spherical harmonics, it can be shown that

$$I_0(\mathbf{r}) = 2(\pi)^{1/2}A_0^0(\mathbf{r}) \qquad (14a)$$

$$I_{2x_ix_j}(\mathbf{r}) = \tfrac{2}{3}(\pi)^{1/2}A_0^0(\mathbf{r})\delta_{ij} \qquad (14b)$$

where δ_{ij} is the Kronecker delta. It follows from Eqs. (14) that

$$I_{2x_ix_j}(\mathbf{r}) = [I_0(\mathbf{r})/3]\delta_{ij} \qquad (15)$$

In view of Eqs. (11) and (15), the radiation pressure tensor is

$$p_{ij}(\mathbf{r}) = [u_r(\mathbf{r})/3]\delta_{ij} \qquad (16)$$

Comparing Eq. (16) with Eq. (10), it follows that the first approximation in effect assumes that the quantities p_{ij} and u_r are related in such a way as if the gas were in local radiation equilibrium. Making use of relations (11b) and (15) in Eqs. (12a) and (12b), we can now replace the exact transport equation by the following equations in the first approximation‡:

$$\partial q_j/\partial x_j = -\alpha I_0 + 4\alpha\sigma T^4 \qquad (17a)$$

$$\partial I_0/\partial x_i = -3\alpha q_i \qquad (i = 1,2,3) \qquad (17b)$$

If I_0 is eliminated from these equations, we obtain the following approximate transport equation in terms of q_i:

$$\frac{\partial}{\partial x_i}\left[\frac{1}{\alpha}\frac{\partial q_j}{\partial x_j}\right] - 3\alpha q_i - 16\sigma T^3\frac{\partial T}{\partial x_i} = 0 \qquad (i = 1,2,3) \qquad (18)$$

‡ These equations can be obtained by another approach known as the spherical harmonic method.[9]

Note that Eq. (18) reproduces the well-known thick-gas (due to Rosseland[10]) and thin-gas approximations when α is large or small. Equations (18) or (17), together with the hydrodynamic equations (1–5), constitute a determinate set of purely differential equations for radiating gas flow.

Linearized Equations for Two-Dimensional Steady Flow

We now obtain the linearized approximation to the foregoing equations. To this end, the assumption is made that disturbances, measured from some reference condition that is in radiation equilibrium, are small. If the disturbance quantities are denoted by primes and the subscript zero denotes the equilibrium reference condition, the dependent variables are then given by $u_1 = U_0 + u_1'$, $u_2 = u_2'$, $p = p_0 + p'$, $\alpha = \alpha_0 + \alpha'$, $q_i = q_i'$, etc. Substituting these variables into the governing equations (1–5) and (18) and proceeding with the linearization in the usual fashion, we obtain for two-dimensional steady flow

$$U_0\frac{\partial\rho'}{\partial x_1} + \rho_0\frac{\partial u_j'}{\partial x_j} = 0 \qquad (19)$$

$$\rho_0U_0\frac{\partial u_i'}{\partial x_1} + \frac{\partial p'}{\partial x_1} = 0 \qquad (i = 1,2) \qquad (20)$$

$$U_0\left(\rho_0\frac{\partial h'}{\partial x_1} - \frac{\partial p'}{\partial x_1}\right) + \frac{\partial q_j'}{\partial x_j} = 0 \qquad (21)$$

$$T' = \frac{1}{R}\left(\frac{p'}{\rho_0} - \frac{p_0}{\rho_0^2}\rho'\right) \qquad (22)$$

$$h' = \frac{\gamma}{\gamma-1}\left(\frac{p'}{\rho_0} - \frac{p_0\rho'}{\rho_0^2}\right) \qquad (23)$$

$$\frac{\partial}{\partial x_i}\left[\frac{\partial q_j'}{\partial x_j}\right] - 3\alpha_0^2q_i' - 16\sigma T_0^3\alpha_0\frac{\partial T'}{\partial x_i} = 0 \qquad (i = 1,2) \qquad (24)$$

We now introduce a perturbation velocity potential ϕ such that $u_i' = \partial\phi/\partial x_i$. Equations (19–24) can then be reduced to a single equation in terms of ϕ. For this purpose, we first differentiate Eq. (22) with respect to x_1 and make use of Eqs. (19) and (20) with $i = 1$ to obtain

$$\frac{\partial T'}{\partial x_1} = \frac{U_0}{\gamma RM_0^2}\left[(1-\gamma M_0^2)\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2}\right] \qquad (25)$$

where M_0 is the Mach number based on the reference equilibrium condition. Eliminating h between Eqs. (21) and (23), we have, with the aid of Eqs. (25) and (20) with $i = 1$,

$$(1-M_0^2)\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{(\gamma-1)M_0^2}{\rho_0U_0^2}\frac{\partial q_j'}{\partial x_j} = 0 \qquad (26)$$

By differentiating Eq. (26) with respect to x_i and using Eqs. (24), we also obtain

$$q_i' = -\frac{\rho_0U_0^2}{3\alpha_0^2(\gamma-1)M_0^2}\frac{\partial L_s}{\partial x_i} - \frac{16\sigma T_0^3}{3\alpha_0}\frac{\partial T'}{\partial x_i} \qquad (i = 1,2) \qquad (27)$$

where

$$L_s \equiv (1-M_0^2)\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2}$$

Substituting Eqs. (27) into Eq. (24) with $i = 1$, we obtain finally, with the aid of relation (25),

$$\frac{\partial}{\partial x_1}\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right]L_s + K\alpha_0\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right]L_T - 3\alpha_0^2\frac{\partial L_s}{\partial x_1} = 0 \qquad (28)$$

where

$$K \equiv \frac{16}{N_{Bo}} \qquad L_T \equiv (1 - \gamma M_0^2)\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2}$$

$$N_{Bo} \equiv \frac{\gamma R \rho_0 U_0}{(\gamma - 1)\sigma T_0^3}$$

which is called the Boltzmann number.

If we proceed in similar fashion from the one-dimensional unsteady-flow equations, it can be shown that the linearized equation with a coordinate system fixed relative to the undisturbed fluid is§

$$\frac{\partial^3 W_s}{\partial x^2 \partial t} + k\alpha_0 \frac{\partial^2 W_T}{\partial x^2} - 3\alpha_0^2 \frac{\partial W_s}{\partial t} = 0 \tag{29}$$

where

$$W_s \equiv \frac{\partial^2 \phi}{\partial t^2} - a_0^2 \frac{\partial^2 \phi}{\partial x^2} \qquad W_T \equiv \frac{\partial^2 \phi}{\partial t^2} - \frac{a_0^2}{\gamma}\frac{\partial^2 \phi}{\partial x^2}$$

$$k \equiv \frac{16\gamma a_0}{N_{Bo}}$$

a_0 is the isentropic speed of sound and $N_{Bo} \equiv [\gamma R \rho_0 a_0/(\gamma - 1)\sigma T_0^3]$.

Equations (28) and (29) are the governing differential equations in terms of the velocity potential for two-dimensional steady flows (coordinate system fixed in the body) and one-dimensional unsteady flows (coordinate system fixed in space) with a small departure from radiation equilibrium. It is seen that they are both of fifth order. Unlike the situation in classical equilibrium-flow theory, however, the structure of these two equations is inherently different. This is as would be expected because of the directional properties of radiation intensity. The relationship of the equations to those of classical theory is apparent: In the limits of a very cold gas ($N_{Bo} \to \infty$), a transparent gas ($\alpha_0 \to 0$), or an opaque gas ($\alpha_0 \to \infty$), Eqs. (28) and (29) reduce to the classical equations. For the limiting case of a very hot gas ($N_{Bo} \to 0$), they also reduce to the classical equations but with the isentropic speed of sound replaced by the isothermal speed of sound. The author has been able to apply the linearized equations (29) and (28) to problems of wave propagation and flow over a wavy wall. The results will be presented at a later time.

References

1 Sen, H. K. and Guess, A. W., "Radiation effects in shock wave structure," Phys. Rev. **108**, 560–564 (1957).

2 Marshak, R. E., "Effect of radiation on shock wave behavior," Phys. Fluids **1**, 24–29 (1958).

3 Zhigulev, V. N., Romishevskii, Ye. A., and Vertushkin, V. K., "Role of radiation in modern gasdynamics," AIAA J. **1**, 1473–1485 (1963).

4 Vincenti, W. G. and Baldwin, B. S., Jr., "Effect of thermal radiation on the propagation of plane acoustic waves," J. Fluid Mech. **12**, 449–477 (1962).

5 Eddington, A., *The Internal Constitution of the Stars* (Cambridge University Press, London, 1926), Chap. V, p. 97.

6 Davison, B., *Neutron Transport Theory* (Oxford University Press, London, 1958), Chap. XII, p. 157.

7 Traugott, S. C., "A differential approximation for radiative transfer with application to normal shock structure," The Martin Co. Res. Rept., RR-34 (1962).

8 Mitchner, M. and Vinokur, M., "Radiation smoothing of shocks with and without a magnetic field," Phys. Fluids **6**, 1682–1692 (1963).

9 Krook, M., "On the solution of equations of transfer," Astrophys. J. **122**, 488–497 (1955).

10 Rosseland, S., *Theoretical Astrophysics* (Oxford University Press, London, 1936), Chap. IX, p. 105.

11 Lick, W. J., "The propagation of small disturbances in a radiating gas," J. Fluid Mech. **18**, 274–285 (1964).

§ During the writing of this note, the author came upon a recent paper by Lick.[11] By using the exponential approximation applicable in the one-dimensional case, Lick also obtained Eq. (29) for one-dimensional flow with radiation. This is as would be expected, since the exponential approximation and the moment method applied to one-dimensional problems are mathematically equivalent.

Evaporation Coefficients from Exposure of a Solid to Laser Radiation

S. S. Penner*

University of California at San Diego, La Jolla, Calif.

THE theoretical calculation of ablation rates in an intense radiation field is generally a complicated problem, because it involves the interaction of various relaxation phenomena (melting, sublimation or evaporation, ionization), effusive flow or "explosive blowoff" in a complex geometry, and radiative energy transfer. However, under suitable exposure to a laser beam, these processes are effectively decoupled, and a simple phenomenological description is possible for the expected linear regression rate.

Typical relaxation times in the solid phase are of the order of the reciprocal frequency of an equivalent Einstein or Debye oscillator, i.e., of the order of 3×10^{-13} sec. These time estimates indicate that a meaningful temperature can be defined, within a volume element containing very many atoms, on exposure to an intense laser beam with duration longer than about 10^{-9} sec.

For a subliming solid, the linear regression rate (dx/dt) is given by the expression[1]

$$dx/dt = j_e/n^{1/3} \leq (p_s/\rho)(m/2\pi kT)^{1/2} \tag{1}$$

where $j_e \simeq \nu \exp(-\Delta h/kT)$ is the evaporation or sublimation frequency, n is the number of atoms or molecules per unit volume in the solid phase (i.e., $n = \rho/m$), p_s is the saturation vapor pressure at the temperature T of the solid with density ρ, m is the mass per atom, k stands for the Boltzmann constant, ν is the frequency of the Einstein oscillator corresponding to the solid, and Δh represents the heat of evaporation or sublimation per atom or molecule. The upper bound in Eq. (1) corresponds to the limit set by the Knudsen equation with unit evaporation coefficient $\mathcal{K}$. This bound may be made explicit by assuming the validity of Trouton's rule. Thus $p_s \simeq p_0 \exp(-\Delta h/kT)$, where the constant p_0 may be evaluated conveniently by assuming that $p_s = 1$ atm at the normal boiling point $T = T_b$. Hence, $p_s \simeq 2.2 \times 10^{10} \exp(-\Delta h/kT)$ dyne/cm^2, where we have set $\Delta h/kT_b = 10$, and

$$\nu/n^{1/3} \leq 2.2 \times 10^{10}(1/\rho)(m/2\pi kT)^{1/2}\exp(-\Delta h/kT)$$

Representative estimates of the Knudsen limits (with an evaporation coefficient of unity) for the linear regression rates of metals and stable compounds near the normal boiling points lead to values of a few centimeters per second, i.e., evaporation is unimportant during radiant-energy input for laser pulses with a duration of less than about 10^{-3} sec.

The evaporation coefficient may be defined by the relation

$$\mathcal{K} \equiv \frac{(dx/dt)}{(p_s/\rho)(m/2\pi kT)^{1/2}} \equiv \frac{[\alpha \exp(-\Delta h/kT)]/n^{1/3}}{(p_s/\rho)(m/2\pi kT)^{1/2}} \tag{2}$$

Received May 25, 1964. This work was supported by the U. S. Army Research Office (Durham) under AROD Grant DA-ARO(D)-31-124-G504.

* Professor of Engineering Physics and Astronautics; also Chairman, Department of Aerospace and Mechanical Engineering Sciences. Associate Fellow Member AIAA.